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A Derivation of Hill's Equation by a Direct Substitution.

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(*Communicated by the Secretaries.*)

The method and most of the notation is that of Hill's paper on the "Motion of the Lunar Perigee," *Acta Mathematica*, vol. viii.

The equations satisfied by u and s are

$$\left.\begin{aligned} D^2u + 2mDu + 2\frac{\partial\Omega}{\partial s} &= 0 \\ D^2s - 2mDs + 2\frac{\partial\Omega}{\partial u} &= 0 \end{aligned}\right\} \quad \dots \quad \dots \quad (1)$$

which lead to Jacobi's Integral

$$DuDs + 2\Omega = 2C \quad \dots \quad \dots \quad \dots \quad (2)$$

We have if ψ be the angle made by the tangent with the axis of x

$$Du = \zeta\frac{d}{d\zeta}(x+is) = e^{i\psi}\zeta\frac{d\sigma}{d\zeta}, \quad Ds = e^{-\psi}\zeta\frac{d\sigma}{d\zeta}$$

where σ is the arc of the curve ; or if ϖ denote $Du \,.\, Ds$,

$$Du = e^{i\psi}\sqrt{\varpi}, \quad Ds = e^{-i\psi}\sqrt{\varpi}$$

Hence $2i\psi = \log(Du/Ds)$; and

$$2iD\psi = \frac{D^2u}{Du} - \frac{D^2s}{Ds} \quad \dots \quad \dots \quad \dots \quad (3)$$

This gives $2i\varpi D\psi = D^2uDs - D^2sDu$, and on substitution from (1)

$$\begin{aligned} 2i\varpi D\psi &= -Ds\{2mDu + 2\Omega_s\} - Du\{2mDs - 2\Omega_u\} \\ &= -4m\varpi - 2\Omega_sDs + 2\Omega_uDu \end{aligned}$$

Hence

$$\Omega_uDu - \Omega_sDs = \varpi(2m + iD\psi) \quad \dots \quad \dots \quad (4)$$

Differentiating (4)

$$\begin{aligned} \Omega_uD^2u - \Omega_sD^2s + (\Omega_{uu}Du + \Omega_{us}Ds)Du - (\Omega_{us}Du + \Omega_{ss}Ds)Ds \\ = (2m + iD\psi)D\varpi + \varpi iD^2\psi \end{aligned}$$

or substituting for Ω_u and Ω_s from (1)

$$\begin{aligned} m(DuD^2s + DsD^2u) + \Omega_{uu}\overline{Du}^2 - \Omega_{ss}\overline{Ds}^2 = (2m + iD\psi)\,D\varpi \\ + \varpi iD^2\psi \end{aligned}$$

giving the equation

$$(i\varpi D^2\psi + \Omega_{ss}\overline{Ds}^2 - \Omega_{uu}\overline{Du}^2) = -(m + iD\psi)D\varpi \quad \dots \quad (5)$$

which will be useful later.

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Let the point on the variational orbit receive an arbitrary tangential displacement $\delta\mathrm{T}$ and a normal displacement $\delta\mathrm{N}$; then the corresponding values of δu, δs are respectively $(\delta\mathrm{T}-i\delta\mathrm{N})e^{i\psi}$ and $(\delta\mathrm{T}+i\delta\mathrm{N})e^{-i\psi}$.

Now subject the Jacobi Integral to an arbitrary variation,

$$\mathrm{D}s\mathrm{D}\delta u+\mathrm{D}u\mathrm{D}\delta s+2\,\Omega_u\delta u+2\,\Omega_s\delta s=0$$

or with the values for $\mathrm{D}u$, $\mathrm{D}s$ above

$$\begin{aligned}&e^{-i\psi}\sqrt{\varpi}e^{i\psi}\{\mathrm{D}(\delta\mathrm{T}-i\delta\mathrm{N})+(\delta\mathrm{T}-i\delta\mathrm{N})i\mathrm{D}\psi\}\\&+\sqrt{\varpi}\quad\{\mathrm{D}(\delta\mathrm{T}+i\delta\mathrm{N})-(\delta\mathrm{T}+i\delta\mathrm{N})i\mathrm{D}\psi\}\\&+2\,\Omega_u(\delta\mathrm{T}-i\delta\mathrm{N})\frac{\mathrm{D}u}{\sqrt{\varpi}}+2\,\Omega_s(\delta\mathrm{T}+i\delta\mathrm{N})\frac{\mathrm{D}s}{\sqrt{\varpi}}=0\end{aligned}$$

whence

$$\varpi\mathrm{D}\delta\mathrm{T}+\delta\mathrm{T}\,.\,\mathrm{D}\Omega+\varpi\mathrm{D}\psi\,.\,\delta\mathrm{N}+(\Omega_s\mathrm{D}s-\Omega_u\mathrm{D}u)i\delta\mathrm{N}=0$$

On substitution from (2) and (4) this gives

$$[\varpi\mathrm{D}-\tfrac{1}{2}\mathrm{D}\varpi]\delta\mathrm{T}=2\varpi(m+i\mathrm{D}\psi)i\delta\mathrm{N}\quad\ldots\quad\ldots\quad(6)$$

We then vary the first equation of motion and obtain

$$\mathrm{D}^2\delta u+2m\mathrm{D}\delta u+2\,\Omega_{us}\delta u+2\,\Omega_{ss}\delta s=0$$

or on substitution

$$\begin{aligned}&e^{i\psi}[\mathrm{D}^2(\delta\mathrm{T}-i\delta\mathrm{N})+2(m+i\mathrm{D}\psi)\mathrm{D}(\delta\mathrm{T}-i\delta\mathrm{N})\\&\quad+(i\mathrm{D}^2\psi+2mi\mathrm{D}\psi-\overline{\mathrm{D}\psi}|^2)(\delta\mathrm{T}-i\delta\mathrm{N})]\\&\quad+2\,\Omega_{us}e^{i\psi}(\delta\mathrm{T}-i\delta\mathrm{N})+2\,\Omega_{ss}e^{-i\psi}(\delta\mathrm{T}+i\delta\mathrm{N})=0\end{aligned}$$

and the equation obtained by varying the second equation of motion differs only in having the signs of i and m changed and Ω_{uu} in place of Ω_{ss}. Multiply these two equations by $e^{-i\psi}$, e^{i} and subtract; we obtain

$$\begin{aligned}2(m+i\mathrm{D}\psi)\mathrm{D}\delta\mathrm{T}&+(i\mathrm{D}^2\psi+e^{-2i\psi}\Omega_{ss}-e^{2i\psi}\Omega_{uu})\delta\mathrm{T}\\&=\mathrm{D}^2i\delta\mathrm{N}+(2mi\mathrm{D}\psi-\overline{\mathrm{D}\psi}|^2)i\delta\mathrm{N}\\&\quad+(2\,\Omega_{us}-e^{-2i\psi}\Omega_{ss}-e^{2i\psi}\Omega_{uu})i\delta\mathrm{N}.\end{aligned}$$

Putting $(\mathrm{D}s)^2/\varpi$ for $e^{-2i\psi}$ and $(\mathrm{D}u)^2/\varpi$ for $e^{2i\psi}$ the coefficient of $\delta\mathrm{T}$ on the left is $-\frac{1}{\varpi}(m+i\mathrm{D}\psi)\mathrm{D}\varpi$ by (5), and hence substituting from (6) the value of the left-hand side is $4(m+i\mathrm{D}\psi)^2\,i\delta\mathrm{N}$.

The equation satisfied by $\delta\mathrm{N}$ is therefore $\mathrm{D}^2\delta\mathrm{N}=\Theta\delta\mathrm{N}$, where

$$\Theta=m^2+3(m+i\mathrm{D}\psi)^2+e^{2i\psi}\Omega_{uu}-2\,\Omega_{us}+e^{-2i\psi}\Omega_{ss}$$

It is easy to verify by change of variables that the terms involving Ω in this expression for Θ are $-\dfrac{\partial^2\Omega}{\partial\mathrm{N}^2}$.

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We have yet to express $-\dfrac{\partial^2 \Omega}{\partial N^2}$ in terms of the differentials of u and s.

Now $-\tfrac{1}{2}D\varpi = D\Omega = \Omega_u Du + \Omega_s Ds.$

Hence

$$
\begin{aligned}
-\tfrac{1}{2}D^2\varpi &= \Omega_u D^2 u + \Omega_s D^2 s \\
&\quad + \Omega_{uu}(Du)^2 + 2\,\Omega_{us} Du Ds + \Omega_{ss}(Ds)^2 \\
&= -\Omega_u(2mDu + 2\,\Omega_s) + \Omega_s(2mDs - 2\,\Omega_u) \\
&\quad + \varpi\left\{e^{2i\psi}\Omega_{uu} + 2\,\Omega_{us} + e^{-2i\psi}\Omega_{ss}\right\} \\
&= -2m\varpi(2m + iD\psi) - 4\,\Omega_u\Omega_s \\
&\quad + \varpi\left\{e^{2i\psi}\Omega_{uu} + 2\,\Omega_{us} + e^{-2i\psi}\Omega_{ss}\right\}
\end{aligned}
$$

Also

$$
\begin{aligned}
4\,\Omega_u\Omega_s &= \left\{(\Omega_u Du + \Omega_s Ds)^2 - (\Omega_u Du - \Omega_s Ds)^2\right\}/\varpi \\
&= \frac{1}{\varpi}(D\Omega)^2 - \varpi(2m + iD\psi)^2
\end{aligned}
$$

and this gives on substitution

$$
\begin{aligned}
& e^{2i\psi}\Omega_{uu} - 2\,\Omega_{us} + e^{-2i\psi}\Omega_{ss} \\
&\quad = -\frac{1}{2\varpi}D^2\varpi + \frac{1}{4\varpi^2}(D\varpi)^2 + m^2 - (m + iD\psi)^2 - 4\,\Omega_{us} \\
&\quad = -D\left(\frac{D\varpi}{2\varpi}\right) - \left(\frac{D\varpi}{2\varpi}\right)^2 + m^2 - (m + iD\psi)^2 - 4\,\Omega_{us}
\end{aligned}
$$

and since

$$\Omega = \frac{K}{(us)^{\frac{1}{2}}} + \tfrac{3}{8}m^2(u+s)^2,$$

$$\Omega_{us} = \frac{3}{4}\frac{K}{r^3} + \tfrac{3}{4}m^2,$$

whence

$$\Theta = -\left(\frac{K}{r^3} + m^2\right) + 2(m + iD\psi)^2 - D\left(\frac{D\varpi}{2\varpi}\right) - \left(\frac{D\varpi}{2\varpi}\right)^2,$$

which is Hill's form.

On the Construction of Telescopes whose Relative or Absolute Focal Length shall be Invariable at all Temperatures. By F. L. O. Wadsworth.

(*Communicated by the Secretaries*)

In the case of all optical instruments which are designed for the accurate measurement of either relative or absolute positions it is extremely necessary to avoid the necessity for refocussing the instrument for changes of temperature. Such refocussing is likely to produce both errors of measurement and errors of adjustment. The first have already been dealt with in